Uniform in time convergence to BEC for a weakly interacting Bose gas with external potentials

(Joint work with Charlotte Dietze)

Jinyeop Lee

LMU Munich

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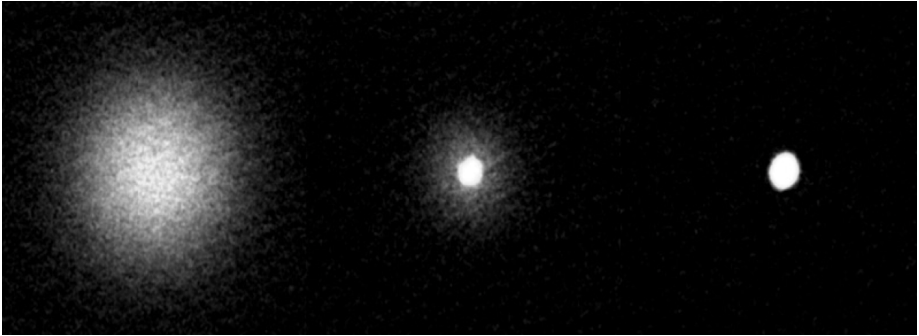
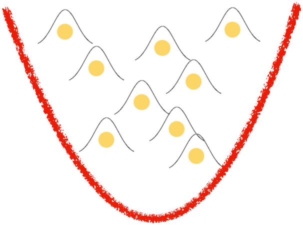
Bose-Einstein condensation



taken by K. Kim from Prof. J.-y. Choi

Our system



$$\Psi_N(x_1, \cdots, x_N)$$

***N*-body mean-field Hamiltonian**

$$H_N = \sum_{j=1}^{N} \left(-\Delta_{x_j} + U(x_j)\right) + \frac{\lambda}{N} \sum_{i<j}^{N} w_N\left(x_i - x_j\right)$$

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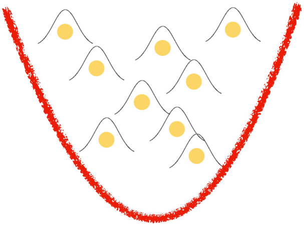

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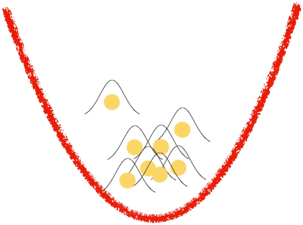

$\Psi_N(x_1, \cdots, x_N)$

N-**body mean-field Hamiltonian**

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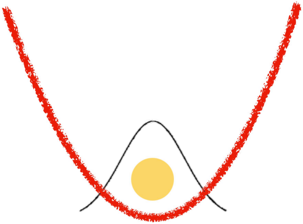
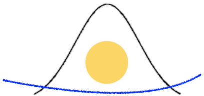


$$\Psi_{N,0}(x_1, \cdots, x_N) \simeq \prod_{j=1}^{N} \varphi_0(x_j)$$

N-body mean-field Hamiltonian

$$H_N = \sum_{j=1}^{N} \left(-\Delta_{x_j} + U(x_j)\right) + \frac{\lambda}{N} \sum_{i<j}^{N} w_N\left(x_i - x_j\right)$$

Our system



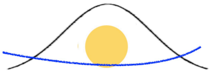
$$\Psi_{N,0}(x_1, \cdots, x_N) = \prod_{j=1}^N \varphi_0(x_j)$$

N-body mean-field Hamiltonian

$$H_N = \sum_{j=1}^N \left(-\Delta_{x_j} + V(x_j)\right) + \frac{\lambda}{N}\sum_{i<j}^N w_N\left(x_i - x_j\right)$$

Our system

$$\Psi_{N,t}(x_1, \cdots, x_N) = e^{-\mathrm{i} H_N t} \psi_{N,0} \simeq ?$$



***N*-body mean-field Hamiltonian**

$$H_N = \sum_{j=1}^{N} \left(-\Delta_{x_j} + V(x_j)\right) + \frac{\lambda}{N} \sum_{i<j}^{N} w_N\left(x_i - x_j\right)$$

Heuristic candidate

Heuristically, if we assume the approximate factorization

$$\psi_{N,t}(\mathbf{x}) \simeq \prod_{j=1}^{N} \varphi_t(x_j) \quad \text{for large } N,$$

then the total potential experienced by the particle x_1 can be approximated by

$$\frac{\lambda}{N} \sum_{j=2}^{N} \int_{\mathbb{R}^3} w_N(x_j - x_1) |\varphi_t(x_j)|^2 \, \mathrm{d}x_j = \lambda(w_N * |\varphi_t|^2)(x_1) .$$

Thus, the evolution of the one-particle wave function φ_t can be described approximately by the nonlinear **Hartree type equation**

$$\mathrm{i}\partial_t \varphi_t = (-\Delta + V)\varphi_t + \lambda(w_N * |\varphi_t|^2)\varphi_t$$

or nonlinear Schrödinger equation

$$\mathrm{i}\partial_t \varphi_t = (-\Delta + V)\varphi_t + \lambda a |\varphi_t|^2 \varphi_t .$$

Natural metric might not be $L^2(\mathbb{R}^{3N})$.

Consider a case in which $\psi = \varphi^{\otimes N}$, $\widetilde{\psi} = (\widetilde{\varphi} \otimes \varphi^{\otimes (N-1)})_s$ such that $\langle \varphi, \widetilde{\varphi} \rangle = 0$, $\|\varphi\|_2 = \|\widetilde{\varphi}\| = 1$, i.e.,

$$\widetilde{\psi}(x_1, \cdots, x_N) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \widetilde{\varphi}(x_j) \prod_{k \neq j}^{N} \varphi(x_k).$$

ψ is very close to $\widetilde{\psi}$. However, for all positive integer N,

$$\|\psi - \widetilde{\psi}\|_2^2 = \|\psi\|_2^2 + \|\widetilde{\psi}\|_2^2 - 2\langle \psi, \widetilde{\psi} \rangle = 2.$$

Now, we want to use the concept of *density operator*.

Marginal density

We introduce **marginal density**

$$\gamma^{(1)}_{N,t}(x_1;y_1) = \int_{\mathbb{R}^{3(N-1)}} \gamma_{N,t}(x_1, z_2, \ldots, z_N; y_1, z_2, \ldots, z_N)\, \mathrm{d}z_2 \ldots \mathrm{d}z_N.$$

Then,

$$\begin{aligned}\gamma(x,y) &= \int_{\mathbb{R}^{3(N-1)}} \overline{\psi(x, z_2, \cdots, z_N)}\psi(y, z_2, \cdots, z_N)\, \mathrm{d}z_2 \ldots \mathrm{d}z_N = \overline{\varphi(x)}\varphi(y),\\ \widetilde{\gamma}(x,y) &= \int_{\mathbb{R}^{3(N-1)}} \overline{\widetilde{\psi}(x, z_2, \cdots, z_N)}\widetilde{\psi}(y, z_2, \cdots, z_N)\, \mathrm{d}z_2 \ldots \mathrm{d}z_N\\ &= \frac{1}{N}\overline{\widetilde{\varphi}(x)}\widetilde{\varphi}(y) + \frac{N-1}{N}\overline{\varphi(x)}\varphi(y).\end{aligned}$$

Now we have $\mathrm{Tr}\,|\gamma - \widetilde{\gamma}| \leq \dfrac{2}{N} \to 0$ as $N \to \infty$.
Thus, γ approximates $\widetilde{\gamma}$ very well for large N in the sense of trace norm.

History

The derivation of the dynamics of the Bose-Einstein condensation has been considered by

- **Hepp** (1974) for $\beta = 0$.
- **Ginibre and Velo** (1979) for $\beta = 0$.
- **Spohn** (1980) for $\beta = 0$.
- **Erdős, Schlein and Yau** (2007,2009,2010) for $0 < \beta \leq 1$.
- **Rodnianski and Schlein** (2009) for $\beta = 0$.
- **Knowles and Pickl** (2010) for $\beta = 0$.
- **Chen, Lee, Schlein** (2011) for $\beta = 0$.
- **Lewin, Nam, and Schlein** (2015) for $\beta = 0$.

Main theorem for $\beta = 0$ (C. Dietze and J. L 2022 preprint)

Suppose the assumptions for the main theorem are satisfied. Let

$$\psi_{N,t} = e^{-\mathrm{i}H_N t}\varphi_0^{\otimes N}, \qquad t \in \mathbb{R}$$

and let φ_t be the solution to the Hartree-type equation

$$\text{(Hartree)} \qquad \begin{cases} \mathrm{i}\partial_t \varphi_t & = (-\Delta + V)\varphi_t + \lambda(w * |\varphi_t|^2)\varphi_t \\ \varphi_t\big|_{t=0} & = \varphi_0. \end{cases}$$

Then there exists a constant $C = C\left(V, \|w\|_1, \|w\|_2\right) > 0$ such that

$$\operatorname{Tr}\left|\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\right| \leq CN^{-1} \qquad \text{for all } t > 0\,.$$

In particular, the constant C does *not* depend on t or N.

Main theorem for $0 < \beta < \frac{1}{3}$ (C. Dietze and J. L 2022 preprint)

Suppose the assumptions for the main theorem are satisfied. Moreover, the **interaction potential** $w \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ is even, real-valued and

$$|w(x)| \leq \frac{C_w}{|x|^\gamma} \qquad \text{for all } |x| \geq 1$$

for some $\gamma > 5$ and $C_w > 0$. Let

$$\psi_{N,t} = e^{-\mathrm{i}H_N t}\psi_{N,0}\,, \qquad t \in \mathbb{R}$$

and let u_t be the solution to the nonlinear Schrödinger equation

(NLS)
$$\begin{cases} \mathrm{i}\partial_t u_t &= (-\Delta + V)u_t + \lambda a|u_t|^2 u_t \\ u_t\big|_{t=0} &= \varphi_0, \end{cases}$$

where $a = \int_{\mathbb{R}^3} \mathrm{d}x\, w(x)$. Then there exists a constant $C > 0$ such that

$$\operatorname{Tr}\left|\gamma^{(1)}_{N,t} - |u_t\rangle\langle u_t|\right| \leq CN^{-\min\left(\frac{1-3\beta}{2},\beta\right)} \qquad \text{for all } t > 0\,.$$

In particular, the constant C does *not* depend on t or N.

Remarks

- A similar result for $\beta = 0$ with a time-dependent constant was proven by **Rodnianski and Schlein**[1]
- A similar result $V = 0$, $\beta = 0$ and $w(x) = \frac{1}{|x|}$ with a time-independent constant was proven.[2].
- Our proof goes along the lines of the proof in previous work[2] and we use a **dispersive estimate** by **Dietze** for the Hartree type equation with an external potential[3].
- **Nam and Napiórkowski**[4] proved a norm approximation with a constant that increases polynomially in t.

[1] I. Rodnianski and B. Schlein. Quantum fluctuations and rate of convergence towards mean field dynamics. Comm. Math. Phys., 291(1):31–61, 2009.

[2] J. L. "On the time dependence of the rate of convergence towards Hartree dynamics for interacting bosons". J. Stat. Phys., 176(2):358–381, 2019.

[3] C. Dietze. "Dispersive Estimates for Nonlinear Schrödinger Equations with External Potentials". J. Math. Phys. 62, 111502 (2021)

[4] P. T. Nam and M. Napiórkowski. "A note on the validity of Bogoliubov correction to mean-field dynamics". J. Math. Pures Appl. 108.5 (2017), pp. 662–688.

Proof strategy

For any $0 \leq \beta < \frac{1}{3}$ let φ_t be the solution to (recall: $w_N(x) := N^{3\beta} w(N^\beta x)$)

$$\begin{cases} \mathrm{i}\partial_t \varphi_t &= (-\Delta + V)\varphi_t + \lambda(w_N * |\varphi_t|^2)\varphi_t \\ \varphi_t\big|_{t=0} &= \varphi_0. \end{cases}$$

We show that

$$\operatorname{Tr}\left|\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\right| \leq \begin{cases} CN^{-1} & \text{if } \beta = 0 \\ CN^{\frac{-1+3\beta}{2}} & \text{if } 0 < \beta < 1/3. \end{cases}$$

This is enough to show the main theorem for $\beta = 0$.

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This is enough to show the main theorem for $\beta = 0$. For $0 < \beta < \frac{1}{3}$, we combine this estimate with the estimate

$$\operatorname{Tr}\Big|\, |\varphi_t\rangle\,\langle\varphi_t| - |u_t\rangle\langle u_t| \Big| \leq 2\|\varphi_t - u_t\|_2 \leq CN^{-\beta}.$$

Key Idea I

Proposition (Dietze, 2021)

Assume $V \in W^{2,\infty}(\mathbb{R}^3)$ to be a 'nice' function. Let $u_0 \in H^2(\mathbb{R}^d)$ and let $u_t \in C\left(\mathbb{R}, H^2(\mathbb{R}^3)\right) \cap C^1\left(\mathbb{R}, H^{-1}(\mathbb{R}^3)\right)$ be the unique global strong solution to the Hartree type equation

$$\begin{cases} \mathrm{i}\partial_t u_t &= (-\Delta + V)u_t + (w * |u_t|^2)u_t \\ u_t\big|_{t=0} &= u_0. \end{cases}$$

Assume that the initial data is sufficiently small, that is,

$$\|e^{\mathrm{i}(-\Delta+V)}u_0\|_1\,, \|u_0\|_{H^2} \leq \varepsilon_0$$

for some $\varepsilon_0 = \varepsilon_0(d, \|V\|_{W^{2,\infty}}, C^V, \|w\|_1) > 0$.
Then there exists a constant $C_0 = C_0(d, \|V\|_{W^{2,\infty}}, C^V, \|w\|_1) \geq 1$ such that

$$\|u_t\|_\infty \leq \frac{C_0}{(1+|t|)^{\frac{3}{2}}} \qquad \text{for all } t \geq 0.$$

Key Idea II

Let J be a compact self-adjoint operator on $L^2(\mathbb{R}^3)$ with $\|J\| = 1$. We want to show

$$\left|\operatorname{Tr}\left(J\big(\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\big)\right)\right| \le \begin{cases} CN^{-1} & \text{if } \beta = 0 \\ CN^{\frac{-1+3\beta}{2}} & \text{if } 0 < \beta < 1/3. \end{cases}$$

The following argument will be used a lot:

Lemma (Grönwall inequality)

$$\frac{d}{dt}\mathcal{U}(t) \le C(t)\mathcal{U}(t) \implies \mathcal{U}(t) \le \mathcal{U}(s)\exp\left(\int_s^t C(t')dt'\right).$$

Remark

In some cases, we face

$$C(t) = CN^{-1/2}\sup_x \|w_N(x-\cdot)\varphi_t\|_{L^2(\mathbb{R}^3)}.$$

What should we do?

$$
\begin{aligned}
C(t) &= CN^{-1/2}\sup_x\left(\int_{\mathbb{R}^3}|w_N(x-y)|^2|\varphi_t(y)|^2\mathrm{d}y\right)^{1/2}\\
&\le CN^{-1/2}\|\varphi_t\|_\infty\sup_x\left(\int_{\mathbb{R}^3}|w_N(x-y)|^2\mathrm{d}y\right)^{1/2}\\
&= CN^{-1/2}\|\varphi_t\|_\infty\|w_N\|_2\\
&= CN^{-1/2}\|\varphi_t\|_\infty\|N^{3\beta/2}w(N^\beta\cdot)\|_2\\
&= CN^{-1+(3\beta/2)}\|\varphi_t\|_\infty\|w\|_2
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathcal{U}(t) &\le \mathcal{U}(0)\exp\left(N^{-1+(3\beta/2)}\int_0^t\|\varphi_s\|_\infty\mathrm{d}s\right)\\
&\le \mathcal{U}(0)\exp\left(N^{-1+(3\beta/2)}\int_0^t\frac{C_0}{(1+s)^{3/2}}\mathrm{d}s\right)\le C\,\mathcal{U}(0).
\end{aligned}
$$

Summary

We showed that for **factorised initial states** $\psi_{N,0}(x_1,\dots,x_N) = \prod_{j=1}^N \varphi_0(x_j)$ evolving according to the time evolution

$$\psi_{N,t} = e^{-\mathrm{i}H_N t}\psi_{N,0}\,, \qquad t \in \mathbb{R}$$

where

$$H_N = \sum_{j=1}^N \big(-\Delta_{x_j} + V(x_j)\big) + \frac{\lambda}{N}\sum_{i<j}^N w_N(x_i - x_j)\,,$$

$|\lambda| \le \lambda_0$ and $0 \le \beta < \frac{1}{3}$, the **one-particle density matrix** $\gamma_{N,t}^{(1)}$ converges **uniformly in time** as $N \to \infty$ to $|\varphi_t\rangle\langle\varphi_t|$ for $\beta = 0$ and to $|u_t\rangle\langle u_t|$ for $0 < \beta < \frac{1}{3}$.

Thank you

Grazie

감사합니다.